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LETTER TO THE EDITOR

Plane gravitational waves colliding with shells of null dust

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Abstract. Using the Szekeres family of solutions of Einstein's equations, exact models describing the collision of plane gravitational waves with planar shells of null dust are constructed.

Dray and 't Hooft [1], using an interesting new technique of matching solutions of the Einstein vacuum equations along a null hypersurface, have recently constructed an exact model of the collision of a pair of planar shells of null dust. Babala [2], on the other hand, has obtained a solution describing the collision between a planar shell of null dust and an impulsive plane gravitational wave. As noted by their authors, the above solutions share several features with the ones describing the collision of plane gravitational waves.

This similarity was to be expected, since a planar shell of null dust can be looked at as the superposition of plane waves with random phases and polarisations. For the same reason, one should be able to arrive at the Dray and t' Hooft and Babala solutions using standard techniques of constructing models of colliding plane gravitational waves, such as the algorithm of Khan and Penrose [3].

In this letter I show that such is indeed the case. This is accomplished by considering those members of the well known Szekeres [4] family of colliding plane-gravitational-wave solutions which admit distribution valued stress-energy tensors with support on null hypersurfaces. I show that this class of solutions contains not only the Dray and 't Hooft and Babala models mentioned above, but models of the collision between planar shells of null dust and gravitational plane waves of all types (shock, impulsive, with a smooth wavefront).

The Szekeres family of solutions is given by the metric [4]

$$ds^2 = 2\,e^{-M}\,dv\,du - e^{-U}(e^{V}\,dx^2 + e^{-V}\,dy^2) \tag{1}$$

where

$$\begin{aligned} M &= [1 - k_1k_2 - \tfrac{1}{4}(k_1 - k_2)^2]\log t + \tfrac{1}{4}k_1^2 \log w + \tfrac{1}{4}k_2^2 \log r + \tfrac{1}{2}k_1k_2 \log(pq - rw) \\ U &= -\log(f+g) \\ V &= k_1 \tanh^{-1}(p/w) + k_2 \tanh^{-1}(q/r) \end{aligned} \tag{2}$$

with

$$\begin{aligned} &p = (\tfrac{1}{2} - f)^{1/2} \qquad q = (\tfrac{1}{2} - g)^{1/2} \qquad r = (\tfrac{1}{2} + f)^{1/2} \qquad w = (\tfrac{1}{2} + g)^{1/2} \\ &t = (f+g)^{1/2} \qquad f = \tfrac{1}{2} - (au)^{n_1}\vartheta(u) \qquad g = \tfrac{1}{2} - (bv)^{n_2}\vartheta(v). \end{aligned} \tag{3}$$

In the above expressions $\vartheta(x)$ represents the Heaviside unit step function and a, b, k_i, and n_i $(i=1,2)$ are real constants, the last two pairs of which satisfy the conditions

$$k_i^2=8(1-1/n_i) \qquad n_i \geqslant 1. \tag{4}$$

The only components of the Ricci tensor which do not vanish everywhere are given by

$$\begin{aligned} R_{uu} &= -U_{,uu}-M_{,u}U_{,u}+\tfrac{1}{2}[(U_{,u})^2+(V_{,u})^2] \\ R_{vv} &= -U_{,vv}-M_{,v}U_{,v}+\tfrac{1}{2}[(U_{,v})^2+(V_{,v})^2] \end{aligned} \tag{5}$$

where $(\ )_{,x} \equiv \vartheta(\ )/\vartheta x$.

Similarly, the only non-vanishing scale invariant Weyl scalars of the Szekeres solution are given by

$$\begin{aligned} \psi_0^0 &= -\tfrac{1}{2}[V_{,vv}-(U_{,v}-M_{,v})V_{,v}] \\ \psi_2^0 &= \tfrac{1}{2}M_{,uv} \\ \psi_4^0 &= -\tfrac{1}{2}[V_{,uu}-(U_{,u}-M_{,u})V_{,u}]. \end{aligned} \tag{6}$$

Let it be noted that ψ_0^0 represents a transverse wave travelling in the u direction, ψ_2^0 is a Coulomb-like component of the gravitational field, and ψ_4^0 corresponds to a transverse wave propagating in the direction of the null coordinate v [5].

Consider, now, the following choices for the parameters k_i and n_i.

Case (i). Colliding planar shells of null dust: $(k_1, k_2)=(0,0)$, $(n_1, n_2)=(1,1)$.

For this set of values of k_i and n_i, equations (1)-(3) give

$$\mathrm{d}s^2=2[1-au\vartheta(u)-bv\vartheta(v)]^{-1/2}\,\mathrm{d}u\,\mathrm{d}v-[1-au\vartheta(u)-bv\vartheta(v)](\mathrm{d}x^2+\mathrm{d}y^2) \tag{7}$$

while (5) and (6) give

$$R_{uu}=-\frac{a\delta(u)}{1-bv\vartheta(v)} \qquad R_{vv}=-\frac{b\delta(v)}{1-au\vartheta(u)} \tag{8}$$

where $\delta(x)$ is the Dirac delta function and

$$\psi_0^0=0 \qquad \psi_2^0=-\frac{ab\vartheta(u)\vartheta(v)}{4[1-au\vartheta(u)-bv\vartheta(v)]^2} \qquad \psi_4^0=0 \tag{9}$$

respectively.

From Einstein's equations and (8), it follows that the only non-vanishing components of the stress-energy tensor are

$$T_{uu}=\frac{a\delta(u)}{1-bv\vartheta(v)} \qquad T_{vv}=\frac{b\delta(v)}{1-au\vartheta(u)}. \tag{10}$$

It is clear from (7), (9) and (10) that, for a and b positive, the above solution represents the collision of a pair of planar shells of null dust having energy density a and b, respectively. In terms of figure 1, the first of the above shells is incident from the right along $u=0$ and collides with the second at $(u, v)=(0,0)$. As a result of the collision, both shells start collapsing isotropically till they end up in a singular state at the 'points' $(u, v)=(0, b^{-1})$ and $(u, v)=(a^{-1}, 0)$, respectively. These points are connected by the singular hypersurface $au+bv=1$ along which $\psi_2^0 \to \infty$.

The solution described above is the Dray and 't Hooft [1] model which was mentioned in the introduction. The way it was derived originally is completely different from the one presented above and the same holds for case (ii) below.

Case (ii). Collision of an impulsive plane gravitational wave with a planar shell of null dust: $(k_1 k_2) = (-2, 0)$, $(n_1, n_2) = (2, 1)$.

By the same method as in the previous case, we find that now

$$ds^2 = (2/w)\, du\, dv - [w - au\vartheta(u)]^2\, dx^2 - [w + au\vartheta(u)]^2\, dy^2 \tag{11}$$

where

$$w(v) = (1 - bv\vartheta(v))^{1/2} \tag{12}$$

$$\psi_0^0 = \frac{abu\vartheta(u)\delta(v)}{2[1-(au)^2]} \qquad \psi_0^4 = (a/w)\delta(u) \qquad \psi_0^2 = 0 \tag{13}$$

$$T_{uu} = 0 \qquad T_{vv} = \frac{b\delta(v)}{1-(au)^2\vartheta(u)}. \tag{14}$$

It follows from (13) and (14) that the planar shell incident from the right in the previous case has now been replaced by an impulsive plane wave. In the present case, spacetime is flat in all regions I, II, III and IV of figure 1, but again both shell and wave end up in a singularity at $(u, v) = (a^{-1}, 0)$ and $(u, v) = (0, b^{-1})$, respectively.

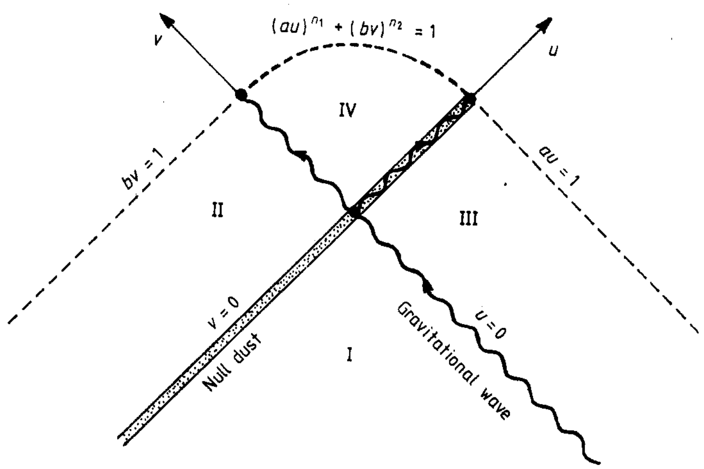

Figure 1. A planar shell of null dust colliding with a plane gravitational wave at $(u, v) = (0, 0)$. The wavy line $u = 0$, $v < 0$ represents the front of the incoming wave.

The above solution was first obtained by Babala [2], who also presented a curious extension of the metric beyond the 'point' $(0, b^{-1})$ and the 'line' $(a^{-1}, 0)$.

Case (iii). Collision of a shock wave with a shell of null dust: $(k_1, k_2) = (-\sqrt{6}, 0)$, $(n_1, n_2) = (4, 1)$.

In this case, $w(v)$ is again given by (12), while

$$p = (au)^2\vartheta(u). \tag{15}$$

Thus

$$\begin{aligned} \psi_0^0 &= \frac{\sqrt{6}\, ba^2u^2\vartheta(u)\delta(v)}{4(1-a^4u^4)} + \frac{\sqrt{6}\, b^2a^6u^6\vartheta(u)\vartheta(v)}{16w^3(w^2-p^2)^2} \\ \psi_4^0 &= \frac{\sqrt{6}\, a^2w^3\vartheta(u)}{(w^2-p^2)^2} \qquad \psi_2^0 = -\frac{ba^4u^3\vartheta(u)\vartheta(v)}{2(w^2-p^2)^2} \end{aligned} \tag{16}$$

and

$$T_{uu}=0 \qquad T_{vv}=\frac{b\delta(v)}{1-(au)^4\vartheta(u)}. \tag{17}$$

It is clear from (16) that now spacetime is flat in regions I and II of figure 1 and that it is a shock wave that approaches the point of collision from the right. As was the case with the collision of the pair of shells, region IV is sealed off towards its future by a singular hypersurface connecting the points $(0, b^{-1})$ and $(a^{-1}, 0)$ and along which $(au)^{n_1}+(bv)^{n_2}=1$.

It should now be obvious that, by keeping $k_2=0$ and varying k_1 appropriately, one can change the character of the gravitational wave colliding with the shell of null dust, as well as the specific nature of the singularity that develops after the collision. A detailed description of such variants of cases (ii) and (iii) considered above will be presented elsewhere, in connection with a one-parameter generalisation of the Szekeres class of solutions obtained recently by Tsoubelis and Wang [6].

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